Definition of the derivative

The definition of the derivative, also called the "difference quotient", is a tool we use to find derivatives "the long way", before we learn all the shortcuts later that let us find them "the fast way".

Mostly it's good to understand the definition of the derivative so that we have a solid foundation for the rest of calculus. So let's talk about how we build the difference quotient.

Secant and tangent lines

A tangent line is a line that *just* barely touches the edge of the graph, intersecting it at only one specific point. Tangent lines look very graceful and tidy.

A secant line, on the other hand, is a line that runs right through the middle of a graph, sometimes hitting it at multiple points, and looks generally meaner.

A tangent line:

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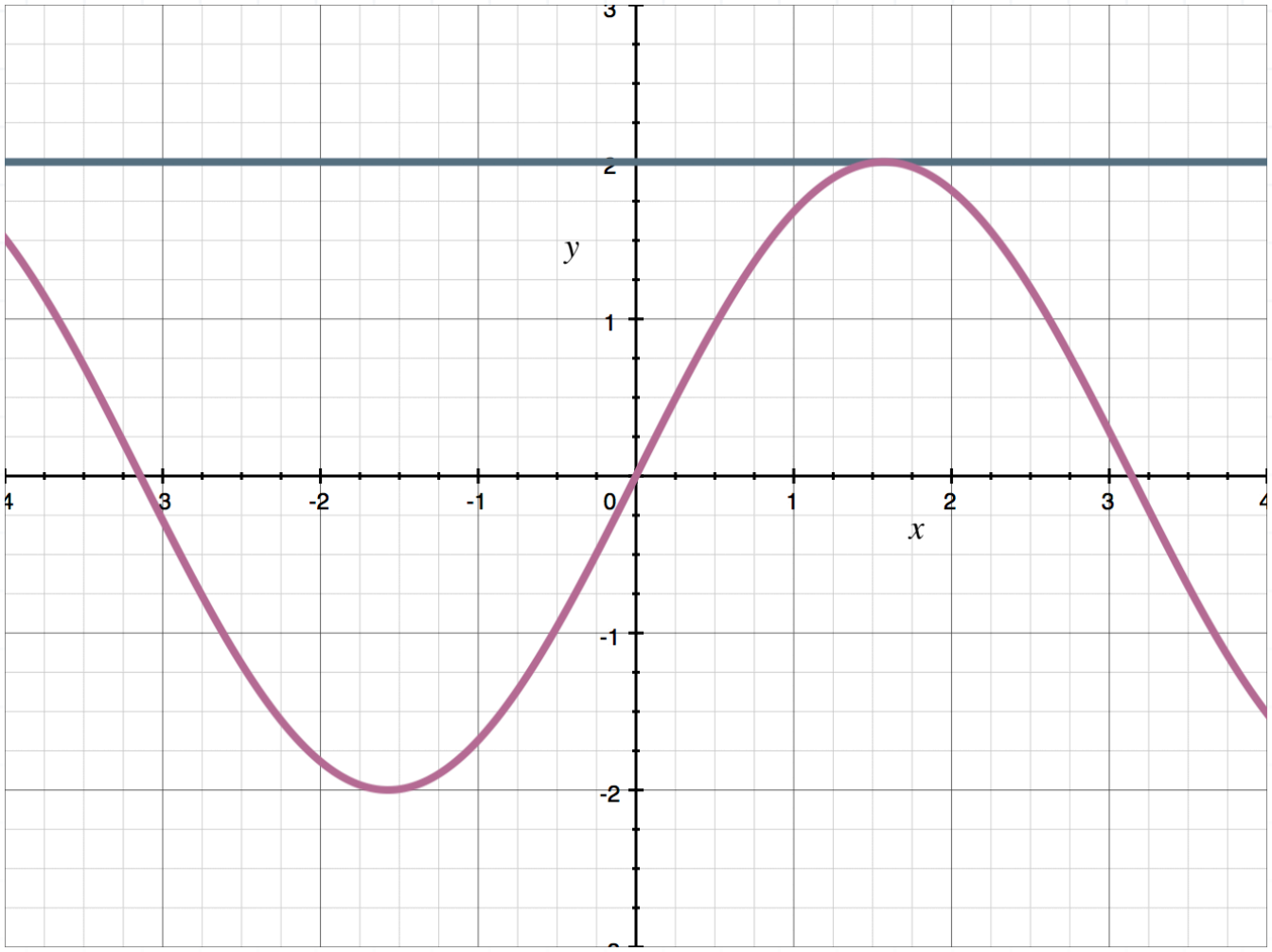
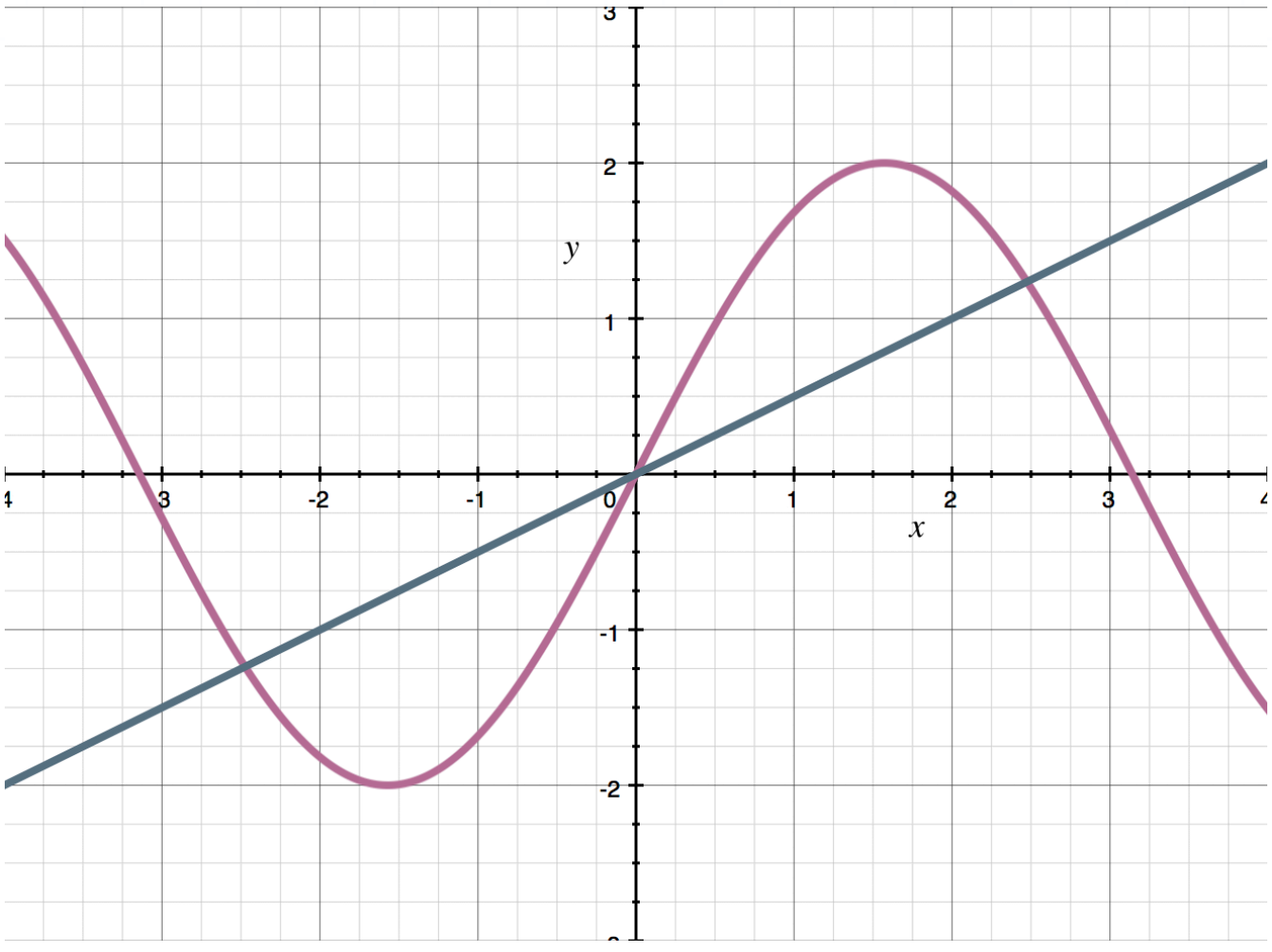


A secant line:



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It's important to realize here that the slope of the secant line is the average rate of change over the interval between the points where the secant line intersects the graph. The slope of the tangent line instead indicates an instantaneous rate of change, or slope, at the single point where it intersects the graph.

Creating the derivative

If we start with a point $(c, f(c))$ on a graph, and move a certain distance Δx to the right of that point, we can call the new point on the graph $(c + \Delta x, f(c + \Delta x))$.

Connecting those points together gives us a secant line, and we can use

$$\frac{y_2 - y_1}{x_2 - x_1}$$

to determine that the slope of the secant line is

$$\frac{f(c + \Delta x) - f(c)}{(c + \Delta x) - c}$$

which when we simplify gives us

$$\frac{f(c + \Delta x) - f(c)}{\Delta x}$$

The point is that, if I take my second point and start moving it slowly left, closer to the original point, the slope of the secant line becomes closer to the slope of the tangent line at the original point.

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In other words, as the secant line moves closer and closer to the tangent line, the points where the line intersects the graph get closer together, which eventually reduces Δx to 0.

Running through this exercise allows us to realize that if I reduce Δx to 0 and the distance between the two secant points becomes nothing, that the slope of the secant line is now exactly the same as the slope of the tangent line. In fact, we've just changed the secant line into the tangent line entirely.

That's how we create the formula above, which is the very definition of the derivative, which is why the definition of the derivative is the slope of the function at a single point.

Using the difference quotient

To find the derivative of a function using the difference quotient, follow these steps:

1. Plug in $x + h$ for every x in your original function. Sometimes you'll also see h as Δx.
2. Plug your answer from Step 1 in for $f(x + h)$ in the difference quotient.
3. Plug your original function in for $f(x)$ in the difference quotient.
4. Put h in the denominator.
5. Expand all terms and collect like terms.

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6. Factor out h from the numerator, then cancel it from the numerator and denominator.

7. Plug in 0 for h and simplify.

Example

Find the derivative.

$$f(x) = x^2 - 5x + 6$$

After replacing x with $(x + \Delta x)$ in $f(x)$, plug in your answer for $f(c + \Delta x)$. Then plug in $f(x)$ as-is for $f(c)$. Put Δx in the denominator.

$$\lim_{\Delta x \to 0} \frac{\left[(x + \Delta x)^2 - 5(x + \Delta x) + 6\right] - (x^2 - 5x + 6)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - 5x - 5\Delta x + 6 - x^2 + 5x - 6}{\Delta x}$$

Collect similar terms together then factor Δx out of the numerator and cancel it from the fraction.

$$\lim_{\Delta x \to 0} \frac{\Delta x^2 + 2x\Delta x - 5\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\Delta x(\Delta x + 2x - 5)}{\Delta x}$$

$$\lim_{\Delta x \to 0} \Delta x + 2x - 5$$

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For Δx, plug in the number you're approaching, in this case 0. Then simplify and solve.

$$0 + 2x - 5$$

$$2x - 5$$

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